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Weyl numbers and eigenvalues of abstract summing operators

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ABSTRACT

We estimate Weyl numbers and eigenvalues of operators via studying their abstract summing norms. In particular we prove estimates of these summing norms for abstract interpolation Lorentz spaces. For this we combine factorization theorems with estimates of concavity constants. Finally we apply our general eigenvalue results to integral operators with kernels of weakly singular type. We obtain asymptotically optimal estimates which extend the well-known classical results.

1. Introduction

The fundamental problem of the theory of Riesz operators in Banach spaces is to determine asymptotic eigenvalue distributions of Riesz operators. We mention that a large number of research articles as well as several monographs are devoted to this topic (see [17,25,5] and the references therein). There are advanced methods to study the mentioned problem, among them is the powerful machinery based on the theories of s-numbers, p-summing operators and local theory of Banach spaces. The axiomatic approach to s-numbers was developed by Pietsch in [25]. The concept of *s-numbers* generalizes the notion of singular numbers in Hilbert spaces to the Banach space setting. The following are particularly important s-numbers, defined for every (bounded linear) operator $T : X \to Y$ between Banach spaces and every $n \in \mathbb{N}$:

- *approximation numbers* $a_n(T) := \inf\{\|T - S\|;\ S \in \mathcal{L}(X, Y),\ \operatorname{rank}(S) < n\}$;
- *Gelfand numbers* $c_n(T) := \inf\{\|T|_G\|;\ G \subset X,\ \operatorname{codim}(G) < n\}$;
- *Weyl numbers* $x_n(T) := \sup\{a_n(TS);\ S \in \mathcal{L}(\ell_2, X),\ \|S\| \leqslant 1\}$.

For compact operators $T : H \to G$ between Hilbert spaces all these numbers are equal to the singular numbers $s_n(T)$. In the context of eigenvalues a central role is played by the Weyl numbers, which were introduced by Pietsch. Combining the Weyl's inequalities in Hilbert spaces with the theory of 2-summing operators, Pietsch [25] extended Weyl's inequalities to the Banach space setting. His remarkable result states that the sequence of eigenvalues of any power-compact operator

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$T : X \to X$ on a complex Banach space satisfies for all $0 < p < \infty$ and $n \in \mathbb{N}$ the inequality

$$\left(\sum_{k=1}^{n}\left|\lambda_k(T)\right|^p\right)^{1/p} \leqslant 2^{1/p}\sqrt{2e}\left(\sum_{k=1}^{n}x_k(T)^p\right)^{1/p}.$$

It is worth mentioning that for every compact operator $T : X \to Y$ between Banach spaces there is a suitable compact operator $S : X \to C[0,1]$ with values in the space $C[0,1]$ of continuous functions over the interval $[0,1]$ such that

$$c_n(S) = c_n(T) \quad \text{and} \quad x_n(S) = x_n(T).$$

Let us note that every bounded linear operator T from a Banach space X to any $C(K)$ space can be represented by a continuous function $f : K \to (X^*, \sigma(X^*, X))$ such that $Tx = \langle x, f\rangle$ for every x in X (see [11, p. 29]). It is proved there that T is weakly compact if and only if $f : K \to (X^*, \sigma(X^*, X^{**}))$ is continuous. It is also shown that T is compact if and only if $f : K \to (X^*, \|\cdot\|_{X^*})$ is continuous.

The mentioned extension of Weyl's inequality motivates the investigation of Weyl numbers of operators between Banach spaces. The key in our study is to estimate abstract summing norms for operators, which generalize p-summing norms. In the case of operators between Banach lattices, the geometry of Banach spaces plays a crucial role, in particular our estimates involve concavity constants with respect to n vectors.

Let us now describe the paper and its contents. In Section 2 we prove some preliminary results. We show estimates of (G, F)-summing norms (called in short, abstract summing norms) of operators. In the case of operators between (quasi)-Banach spaces we use a variant of Krivine's theorem due to Kalton [13]. This allows us to estimate $(G, 2)$-summing norms with respect to n vectors via the $(G, 2)$-concavity constants with respect to n vectors whenever the domain space is a $C(K)$-space.

In Section 3 we obtain one-sided interpolation estimates for (E, p)-summing norms in the setting of abstract Lorentz spaces. This result seems to be of independent interest, due to the fact that it provides new information even in the case of p-summing norms. We transfer this result to a more general setting, namely to the case of intermediate Banach spaces with respect to a compatible couple of Banach spaces. Further, we combine these results with the obtained ones in the previous section. In particular, we prove that for a large class of rearrangement invariant spaces X over a finite non-atomic measure space a sharp estimate $x_n(I : L_\infty \to X) \asymp \psi_X(1/n)$ holds, where ψ_X is the fundamental function of X.

In Section 4 we present some sharp estimates on the concavity constants with respect to n vectors for a special class of rearrangement invariant spaces including e.g. Lorentz spaces $L_{p,q}$ and Zygmund spaces.

Finally, in Section 5, we apply our previous results to integral operators with kernels of weakly singular type. We prove upper estimates for the eigenvalues of such operators which extend the classical results for weakly singular integral operators. Moreover we show that our results are asymptotically optimal.

2. Estimates of Weyl numbers

Let $(\Omega, \mu) := (\Omega, \Sigma, \mu)$ be a complete σ-finite measure space and let X be a Banach space. Throughout the paper $L_0(\mu, X)$ denotes the space of equivalence classes of strongly measurable X-valued functions on Ω, equipped with the topology of convergence in measure (on sets of finite μ-measure). In the case $X = \mathbb{R}$ we write in short $L_0(\mu)$ instead of $L_0(\mu, \mathbb{R})$. By a *Banach lattice* (on (Ω, μ)), we shall mean a Banach space X which is a subspace of $L_0(\mu)$ such that there exists $u \in X$ with $u > 0$ a.e. and if $|x| \leqslant |y|$ a.e., where $y \in X$ and $x \in L_0(\mu)$, then $x \in X$ and $\|x\|_X \leqslant \|y\|_X$. A Banach lattice X is said to be *maximal* if its unit ball $B_X = \{x;\ \|x\|_X \leqslant 1\}$ is a closed subset in $L_0(\mu)$.

A Banach lattice X is said to be *order-continuous* if for every sequence $x_n \downarrow 0$ we have $\|x_n\|_X \to 0$. The Köthe dual X' of a Banach lattice X on (Ω, μ) is defined as

$$X' = \left\{x \in L_0(\mu);\ \|x\|_{X'} := \sup_{\|y\|_X \leqslant 1} \int_\Omega |xy|\, d\mu < \infty\right\}.$$

Note that X' is a Banach lattice under the norm $\|\cdot\|_{X'}$. Moreover, the Köthe dual X' is lattice isometric to a closed subspace of the topological dual X^*.

For a given Banach lattice E on a measure space (Ω, μ) and a Banach space X, we denote by $E(X)$ the set of all $f \in L_0(\mu, X)$ such that $\|f(\cdot)\|_X \in E$. This is a Banach space under pointwise operations and the natural norm

$$\|f\|_{E(X)} := \big\| \big\|f(\cdot)\big\|_X \big\|_E.$$

Let X be a Banach space. For any $f \in L_0(\mu, X^*)$, we define a continuous linear map $T_f : X \to L_0(\mu)$ by

$$T_f x(s) = \big\langle x, f(s)\big\rangle, \quad x \in X,\ s \in \Omega.$$

Following [5], we will call f the (abstract) kernel generating the operator T_f. If E is a Banach lattice on (Ω, μ) and $f \in E(X^*)$, then T_f defines an operator from X into E.

We need some further definitions. Let F and G be Banach sequence spaces with $F \hookrightarrow G$, i.e. F is a linear subspace of G and the embedding is continuous. For every quasi-Banach lattice E and each $n \in \mathbb{N}$ we define

$$M_{G,F}^n(E) = \sup\left\{ \left\| \sum_{k=1}^n \|x_k\|_E e_k \right\|_G ;\ \left\| \left\| \sum_{k=1}^n |x_k| e_k \right\|_F \right\|_E \leqslant 1 \right\}.$$

If $M_{G,F}(E) := \sup_{n\in\mathbb{N}} M_{G,F}^n(E) < \infty$, then E is called (G,F)-concave. For $G = F = \ell_p$ with $1 \leqslant p < \infty$ we obtain the notion of p-concavity (cf. [19]), and instead of $M_{\ell_p,\ell_p}^n(E)$ we write in short $M_{(p)}^n(E)$.

If X is a Banach space and Y a quasi-Banach space, then for every operator $T : X \to Y$ and each $n \in \mathbb{N}$ we define

$$\pi_{G,F}^n(T) = \sup\left\{ \left\| \sum_{k=1}^n \|Tx_k\|_Y e_k \right\|_G ;\ \sup_{\|x^*\|_{X^*}\leqslant 1} \left\| \sum_{k=1}^n \langle x_k, x^*\rangle e_k \right\|_F \leqslant 1 \right\}.$$

If $\pi_{G,F}(T) := \sup_{n\in\mathbb{N}} \pi_{G,F}^n(T) < \infty$, then T is called (G,F)-summing. For $F = \ell_q$ and $G = \ell_p$, with $1 \leqslant q \leqslant p < \infty$, we obtain the well-known notion of absolutely (p,q)-summing operators (cf. [25]). If $F = \ell_2$ and $\ell_2 \hookrightarrow G$, then we get the $(G,2)$-summing operators (see e.g. [9,10]).

Throughout the paper we use the following notation: Given two sequences (a_n) and (b_n) of nonnegative real numbers we write $a_n \prec b_n$ or $a_n = \mathcal{O}(b_n)$, if there is a constant $c > 0$ such that $a_n \leqslant cb_n$ for all $n \in \mathbb{N}$, while $a_n \asymp b_n$ means that $a_n \prec b_n$ and $b_n \prec a_n$ hold. Analogously we use the symbols $f \prec g$ and $f \asymp g$ for nonnegative real functions.

Proposition 2.1. *Assume that F and G are Banach sequence lattices with $F \hookrightarrow G$, let E be a Banach lattice and X a Banach space. If T_f is an integral operator generated by an abstract kernel $f \in E(X^*)$, then we have*

(i) $\pi_{G,F}^n(T_f : X \to E) \leqslant M_{G,F}^n(E)\|f\|_{E(X^*)}$ *for each* $n \in \mathbb{N}$.
(ii) *If E is (G,F)-concave, then $T_f : X \to E$ is (G,F)-summing with*

$$\pi_{G,F}(T_f) \leqslant M_{G,F}(E)\|f\|_{E(X^*)}.$$

Proof. (i) Let the lattice E be defined on the measure space (Ω,μ). For $f \in E(X^*)$, let $\widetilde{\Omega} := \{s \in \Omega;\ f(s) \neq 0\}$. For $s \in \widetilde{\Omega}$ define $a^*(s) \in B_{X^*}$ by

$$a^*(s) = \frac{f(s)}{\|f(s)\|_{X^*}} \chi_{\widetilde{\Omega}}(s).$$

For any $x_1, \dots, x_n \in E$ we have with $C = M_{G,F}^{(n)}(E)$,

$$\begin{aligned} \left\| \sum_{k=1}^n \|T_f x_k\|_E e_k \right\|_G &\leqslant C \left\| \left\| \sum_{k=1}^n \langle x_k, f(\cdot)\rangle e_k \right\|_F \right\|_E \\ &\leqslant C \left\| \|f(\cdot)\|_{X^*} \left\| \sum_{k=1}^n \left|\langle x_k, a^*(\cdot)\rangle\right| e_k \right\|_F \right\|_E \\ &\leqslant C\|f\|_{E(X^*)} \sup_{\|x^*\|_{X^*}\leqslant 1} \left\| \sum_{k=1}^n \langle x_k, x^*\rangle e_k \right\|_F, \end{aligned}$$

and this completes the proof of (i).

The second statement (ii) is an immediate consequence of (i). □

We will need the following result, where we use the notation

$$\lambda_G(n) := \left\| \sum_{k=1}^n e_k \right\|_G$$

for the fundamental function of a Banach sequence space G. Moreover, we set

$$\Lambda_G(n) := \left\| \sum_{k=1}^n \frac{1}{\lambda_G(k)} e_k \right\|_G.$$

Proposition 2.2. *Let G be a Banach sequence space such that $\ell_2 \hookrightarrow G$. Then for every operator $T : X \to Y$ between Banach spaces and every $n \in \mathbb{N}$ we have*

(i) $\lambda_G(n)x_n(T) \leqslant \pi^n_{G,2}(T)$.
(ii) $\|\sum_{k=1}^n x_k(T)e_k\|_G \leqslant \Lambda_G(n)\pi^n_{G,2}(T)$.
(iii) *If G is a symmetric Banach sequence space, then*

$$\left\|\sum_{k=1}^n x_k(T)e_k\right\|_G \leqslant (1+\log_2 n)\pi^n_{G,2}(T).$$

Proof. (i) In [9, Prop. 2] the inequality $\lambda_G(n)x_n(T) \leqslant \pi_{G,2}(T)$ was proved for all $n \in \mathbb{N}$ and every $\pi_{G,2}$-summing operator T, but the proof even gives the same inequality for *arbitrary* operators T with $\pi_{G,2}(T)$ replaced by $\pi^n_{G,2}(T)$, which is (i).

(ii) This is an obvious consequence of (i).

(iii) In view of (ii) it is enough to prove $\Lambda_G(n) \leqslant (1+\log_2 n)\lambda_G(n)$. Given any natural number n, we determine $m \in \mathbb{N}_0$ such that $2^m \leqslant n < 2^{m+1}$; and for $j = 0,\dots,m$ we consider the index sets $I_j = \{k \in \mathbb{N}:\ 2^j \leqslant k < 2^{j+1}\}$. Using the symmetry of the norm in G, we get

$$\left\|\sum_{k\in I_j} e_k\right\|_G = \lambda_G(\operatorname{card} I_j) = \lambda_G\bigl(2^j\bigr).$$

Now the lattice property of G and the triangle inequality easily imply the desired estimate,

$$\begin{aligned}\Lambda_G(n) &= \left\|\sum_{k=1}^n \frac{1}{\lambda_G(k)}e_k\right\|_G \leqslant \sum_{j=0}^m \left\|\sum_{k\in I_j}\frac{1}{\lambda_G(k)}e_k\right\|_G \\ &\leqslant \sum_{j=0}^m \frac{1}{\lambda_G(2^j)}\left\|\sum_{k\in I_j} e_k\right\|_G = m+1 \leqslant 1+\log_2 n. \qquad \square\end{aligned}$$

Before proving the next result, we need to recall a result of Kalton [13] for L-convex quasi-Banach lattices, which extends Krivine's theorem on operators between Banach lattices (see, e.g. [26]). Following Kalton [13], a quasi-Banach lattice X is said to be L-convex if there exists $0 < \varepsilon < 1$ so that if $u \in X_+$, with $\|u\| = 1$ and $0 \leqslant x_k \leqslant u$ $(k = 1,\dots,n)$ satisfy $(x_1 + \cdots + x_n)/n \geqslant (1-\varepsilon)u$, then $\max_{1\leqslant k\leqslant n}\|x_k\| \geqslant \varepsilon$.

Theorem 2.3. *For every L-convex quasi-Banach lattice Y there is a constant $A > 0$ with the following property: If X is a quasi-Banach lattice and $T : X \to Y$ a bounded linear operator, then for all $n \in \mathbb{N}$ and arbitrary vectors $x_1,\dots,x_n \in X$ the inequality*

$$\left\|\left(\sum_{k=1}^n |Tx_k|^2\right)^{1/2}\right\|_Y \leqslant A\|T\|\left\|\left(\sum_{k=1}^n |x_k|^2\right)^{1/2}\right\|_X$$

holds.

Remark. We note that in case of Banach spaces Krivine's result is true with a universal constant $A = K_G$, where K_G is the Grothendieck constant.

As an application of the above we have the following.

Theorem 2.4. *Let G be a quasi-Banach sequence lattice such that $\ell_2 \hookrightarrow G$, and let Y be an L-convex quasi-Banach lattice. Then for every operator $T : C(K) \to Y$ there exists a constant A depending only on Y so that for each $n \in \mathbb{N}$ we have*

$$\pi^n_{G,2}\bigl(T : C(K) \to Y\bigr) \leqslant A\|T\|M^n_{G,2}(Y).$$

In particular, if Y is a Banach lattice and $G = \ell_2$ we have

$$\pi^n_2\bigl(T : C(K) \to Y\bigr) \leqslant K_G\|T\|M^n_{(2)}(Y).$$

Proof. Let $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in C(K)$ be given. Applying Theorem 2.3 we obtain

$$\begin{aligned}\left\| \sum_{k=1}^{n} \|Tx_k\|_Y e_k \right\|_G &\leqslant M_{G,2}^n(Y) \left\| \left(\sum_{k=1}^{n} |Tx_k|^2 \right)^{1/2} \right\|_Y \\ &\leqslant A\|T\| M_{G,2}^n(Y) \left\| \left(\sum_{k=1}^{n} |x_k|^2 \right)^{1/2} \right\|_{C(K)} \\ &= A\|T\| M_{G,2}^n(Y) \sup_{\|x^*\|_{C(K)^*} \leqslant 1} \left(\sum_{k=1}^{n} \left|x^*(x_k)\right|^2 \right)^{1/2},\end{aligned}$$

and this gives the required estimate. □

Now the following corollary is clear. It is a variant of the result [23, Theorem 1] in the setting of quasi-Banach lattices.

Corollary 2.5. *Let G be a quasi-Banach sequence lattice such that $\ell_2 \hookrightarrow G$, and let Y be an L-convex quasi-Banach lattice. Then for every operator $T : C(K) \to Y$ we have*

(i) *T is $(G, 2)$-summing whenever Y is $(G, 2)$-concave.*
(ii) *T is $(q, 2)$-summing whenever Y is q-concave with $2 \leqslant q < \infty$.*

3. Interpolation of abstract summing norms

In the local theory of Banach spaces the concept of summing operators is of special interest. In this section we use interpolation to show general estimates of (E, p)-summing norms with respect to n vectors. We apply these results to abstract Lorentz spaces and rearrangement invariant spaces. We need to recall some basic definitions.

In what follows the set of all functions $\varphi : [0, \infty) \times [0, \infty) \to [0, \infty)$ which are non-decreasing in each variable and positively homogeneous (that is, $\varphi(\lambda s, \lambda t) = \lambda\varphi(s, t)$ for all $\lambda, s, t \geqslant 0$) is denoted by Φ.

For $\varphi \in \Phi$ and $s, t \geqslant 0$ we define

$$\overline{\varphi}(s, t) = \sup\left\{ \frac{\varphi(us, vt)}{\varphi(u, v)};\ u, v > 0 \right\}.$$

Following [2], a mapping $\mathcal{F}$ acting on the category of all compatible Banach couples is called an *interpolation functor*, if for every couple $\overline{X} = (X_0, X_1)$ the Banach space $\mathcal{F}(\overline{X})$ is intermediate with respect to $\overline{X}$ (i.e., $X_0 \cap X_1 \subset \mathcal{F}(\overline{X}) \subset X_0 + X_1$), and $T : \mathcal{F}(\overline{X}) \to \mathcal{F}(\overline{Y})$ is bounded for each $T : \overline{X} \to \overline{Y}$ (meaning that $T : X_0 + X_1 \to Y_0 + Y_1$ is linear and its restrictions $T : X_j \to Y_j$, $j = 0, 1$, are defined and bounded). If additionally there is a constant $C > 0$ such that for each $T : \overline{X} \to \overline{Y}$

$$\left\|T : \mathcal{F}(\overline{X}) \to \mathcal{F}(\overline{Y})\right\| \leqslant C \max\left\{\|T : X_0 \to Y_0\|, \|T : X_1 \to Y_1\|\right\},$$

then $\mathcal{F}$ is called *bounded*. Clearly we always have $C \geqslant 1$, and if $C = 1$ then $\mathcal{F}$ is called *exact*. For a bounded interpolation functor $\mathcal{F}$ we define the *fundamental function* $\psi_{\mathcal{F}}$ of $\mathcal{F}$ by

$$\psi_{\mathcal{F}}(s, t) = \sup\left\|T : \mathcal{F}(\overline{X}) \to \mathcal{F}(\overline{Y})\right\|,$$

where the supremum is taken over all Banach couples $\overline{X}$, $\overline{Y}$ and all operators $T : \overline{X} \to \overline{Y}$ such that $\|T : X_0 \to Y_0\| \leqslant s$ and $\|T : X_1 \to Y_1\| \leqslant t$. From the definition it immediately follows that we have for all couples $\overline{X}$, $\overline{Y}$ and all $T : \overline{X} \to \overline{Y}$ the inequality

$$\left\|T : \mathcal{F}(\overline{X}) \to \mathcal{F}(\overline{Y})\right\| \leqslant \psi_{\mathcal{F}}\left(\|T : X_0 \to Y_0\|, \|T : X_1 \to Y_1\|\right).$$

We will need the following preliminary result which plays a crucial role in the sequel.

Proposition 3.1. *Let $\mathcal{F}$ be an exact interpolation functor, (Y_0, Y_1) a compatible Banach couple, Y an intermediate space with respect to (Y_0, Y_1), and E, E_0, E_1 Banach sequence spaces containing ℓ_p where $1 \leqslant p < \infty$. Define, for each positive integer n,*

$$C_n = \left\|\mathrm{id} : \mathcal{F}\left(E_0^n(Y_0), E_1^n(Y_1)\right) \to E^n(Y)\right\|.$$

Then, for any Banach space X and every $T : (X, X) \to (Y_0, Y_1)$, we have

$$\pi_{E,p}^n(T : X \to Y) \leqslant C_n \psi_{\mathcal{F}}\left(\pi_{E_0,p}^n(T : X \to Y_0), \pi_{E_1,p}^n(T : X \to Y_1)\right).$$

Proof. For each $n \in \mathbb{N}$ we consider the linear operator $\phi_n : \mathcal{L}(\ell_{p'}^n, X) \to \prod_{k=1}^n (Y_0 + Y_1)$ defined by $\phi_n(S) = \{TSe_j\}_{j=1}^n$ for $S \in \mathcal{L}(\ell_{p'}^n, X)$ where $1/p + 1/p' = 1$. Since for $j = 0, 1$ we have

$$\left\|\phi_n : \mathcal{L}\left(\ell_{p'}^n, X\right) \to E_j^n(Y_j)\right\| = \pi_{E_j,p}^n(T : X \to Y_j),$$

it follows by the interpolation property that

$$\left\|\phi_n : \mathcal{L}\left(\ell_{p'}^n, X\right) \to \mathcal{F}\left(E_0^n(Y_0), E_1^n(Y_1)\right)\right\| \leqslant \psi_{\mathcal{F}}\left(\pi_{E_0,p}^n(T : X \to Y_0), \pi_{E_1,p}^n(T : X \to Y_1)\right).$$

Combining the above with our hypothesis implies

$$\begin{aligned}\pi_{E,p}^n(T : X \to Y) &= \left\|\phi_n : \mathcal{L}\left(\ell_{p'}^n, X\right) \to E^n(Y)\right\| \\ &\leqslant C_n \psi_{\mathcal{F}}\left(\pi_{E_0,p}^n(T : X \to Y_0), \pi_{E_1,p}^n(T : X \to Y_1)\right),\end{aligned}$$

which completes the proof. $\square$

We show a general approach which will allow us to use the above proposition in abstract cases. Before the proof of the next result we need some additional notation. Let $\varphi \in \Phi$ and $\overline{X} = (X_0, X_1)$ be a Banach couple. Following [24], the abstract Lorentz space $\Lambda_\varphi(\overline{X})$ consists of all $x \in X_0 + X_1$ such that

$$x = \sum_{n\in\mathbb{Z}} x_n \quad \text{(convergence in } X_0 + X_1\text{)},$$

where $x_n \in X_0 \cap X_1$, and $\sum_{n\in\mathbb{Z}} \varphi(\|x_n\|_{X_0}, \|x_n\|_{X_1}) < \infty$. The norm on $\Lambda_\varphi(\overline{X})$ is defined by

$$\|x\|_{\Lambda_\varphi(\overline{X})} = \inf \sum_{n\in\mathbb{Z}} \varphi\left(\|x_n\|_{X_0}, \|x_n\|_{X_1}\right),$$

where the infimum is taken over all series described above. It is easily verified that Λ_φ is an exact interpolation functor. In fact we have the following simple observation.

Proposition 3.2. *Let $\varphi \in \Phi$. Then the fundamental function of the interpolation functor Λ_φ satisfies*

$$\psi_{\Lambda_\varphi}(s,t) \leqslant \overline{\varphi}(s,t), \quad s, t > 0.$$

Proof. Let $T : \overline{X} \to \overline{Y}$ be an operator between Banach couples with $\|T : X_0 \to Y_0\| \leqslant s$ and $\|T : X_1 \to Y_1\| \leqslant t$. Fix $x \in \Lambda_\varphi(\overline{X})$ with $x = \sum_{n\in\mathbb{Z}} x_n$ (convergence in $X_0 + X_1$) and

$$\sum_{n\in\mathbb{Z}} \varphi\left(\|x_n\|_{X_0}, \|x_n\|_{X_1}\right) < \infty.$$

Then we have $Tx = \sum_{n\in\mathbb{Z}} Tx_n$ (convergence in $Y_0 + Y_1$) and

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \varphi\left(\|Tx_n\|_{Y_0}, \|Tx_n\|_{Y_1}\right) &\leqslant \sum_{n\in\mathbb{Z}} \varphi\left(s\|x_n\|_{X_0}, t\|x_n\|_{X_1}\right) \\ &\leqslant \overline{\varphi}(s,t) \sum_{n\in\mathbb{Z}} \varphi\left(\|x_n\|_{X_0}, \|x_n\|_{X_1}\right).\end{aligned}$$

In consequence, we obtain

$$\|Tx\|_{\Lambda_\varphi(\overline{Y})} \leqslant \overline{\varphi}(s,t)\|x\|_{\Lambda_\varphi(\overline{X})},$$

and this yields the desired inequality. $\square$

We need the following lemma (see [21]).

Lemma 3.3. *Let (X_0, X_1) be an interpolation couple and suppose that E_0 and E_1 are two Banach lattices, both defined on a common measure space (Ω, μ). Then for each $\varphi \in \Phi$,*

$$\Lambda_\varphi\left(E_0(X_0), E_1(X_1)\right) \hookrightarrow \phi(E_0, E_1)\left(\Lambda_\varphi(X_0, X_1)\right),$$

where ϕ is the greatest concave minorant of $\overline{\varphi}$.

Our next result is a simple consequence of Propositions 3.1, 3.2 and Lemma 3.3.

Theorem 3.4. *Let $\varphi \in \Phi$, let (Y_0, Y_1) be a Banach couple, and E_0, E_1 be Banach sequence lattices on $\mathbb{N}$ containing ℓ_p where $1 \leqslant p < \infty$. Then for any Banach space X and every $T : (X, X) \to (Y_0, Y_1)$ and each $n \in \mathbb{N}$, we have*

$$\pi^n_{\phi(E_0,E_1),p}\big(T : X \to \Lambda_\varphi(Y_0, Y_1)\big) \leqslant \phi\big(\pi^n_{E_0,p}(T : X \to Y_0), \pi^n_{E_1,p}(T : X \to Y_1)\big),$$

where ϕ is the greatest concave minorant of $\overline{\varphi}$.

Let (A_0, A_1) be a Banach couple and let A be a Banach space such that $A_0 \cap A_1 \hookrightarrow A$. The *characteristic function of A* with respect to (A_0, A_1) is defined by

$$\varphi(s,t) = \sup\big\{\|a\|_A;\ a \in A_0 \cap A_1,\ \|a\|_{A_0} \leqslant s,\ \|a\|_{A_1} \leqslant t\big\}, \quad \text{for } s, t > 0.$$

The following special case of the preceding theorem seems to be of independent interest from the point of view of applications.

Corollary 3.5. *Let (Y_0, Y_1) be a Banach couple, Y be a Banach space such that $Y_0 \cap Y_1 \hookrightarrow Y$ and let φ be the characteristic function of Y with respect to (Y_0, Y_1). If E is a Banach sequence lattice such that $\ell_p \hookrightarrow E$ for $1 \leqslant p \leqslant \infty$, then for any Banach space X, every operator $T : (X, X) \to (Y_0, Y_1)$ and each $n \in \mathbb{N}$ we have*

$$\pi^n_{\phi(E,\ell_\infty),p}(T : X \to Y) \leqslant \phi\big(\pi^n_{E,p}(T : X \to Y_0), \|T : X \to Y_1\|\big),$$

where ϕ is the greatest concave minorant of $\overline{\varphi}$.

Proof. From the definition of the fundamental function φ of Y with respect to (Y_0, Y_1), it follows that for every $y \in Y_0 \cap Y_1$ we have

$$\|y\|_Y \leqslant \varphi\big(\|y\|_{Y_0}, \|y\|_{Y_1}\big).$$

This implies that the inclusion map $\mathrm{id} : \Lambda_\varphi(Y_0, Y_1) \to Y$ has norm less than or equal to 1. In consequence, for every $T : (X, X) \to (Y_0, Y_1)$, the operator $T : X \to Y$ factors through an abstract Lorentz,

$$T : X \to \Lambda_\varphi(Y_0, Y_1) \xrightarrow{\mathrm{id}} Y.$$

To conclude the proof it is enough to apply Theorem 3.4 to the operator $T : X \to \Lambda_\varphi(Y_0, Y_1)$. □

We show further applications of the above result. Following [24], the function φ which corresponds to an exact interpolation functor $\mathcal{F}$ by the equality

$$\mathcal{F}(s\mathbb{R}, t\mathbb{R}) = \varphi(s,t)\mathbb{R},$$

is called the *characteristic function* of the functor $\mathcal{F}$. Here $\alpha\mathbb{R}$ denotes $\mathbb{R}$ equipped with the norm $\|\cdot\|_{\alpha\mathbb{R}} = \alpha|\cdot|$ for $\alpha > 0$.

Theorem 3.6. *Let $\mathcal{F}$ be an exact interpolation functor with characteristic function φ. If E_j are Banach sequence lattices such that $\ell_p \subset E_j$ for $j = 0, 1$ and $1 \leqslant p < \infty$, then for every operator $T : (X, X) \to (Y_0, Y_1)$ between Banach couples and each $n \in \mathbb{N}$, we have*

$$\pi^n_{\phi(E_0,E_1),p}\big(T : X \to \mathcal{F}(Y_0, Y_1)\big) \leqslant \phi\big(\pi^n_{E_0,p}(T : X \to Y_0), \pi^n_{E_1,p}(T : X \to Y_1)\big),$$

where ϕ is the greatest concave minorant of $\overline{\varphi}$.

Proof. It easy to check (see [24]) that for any Banach couple $\overline{Y} = (Y_0, Y_1)$ we have

$$\|y\|_{\mathcal{F}(\overline{Y})} \leqslant \varphi\big(\|y\|_{Y_0}, \|y\|_{Y_1}\big), \quad y \in Y_0 \cap Y_1.$$

This implies

$$\Lambda_\varphi(Y_0, Y_1) \hookrightarrow \mathcal{F}(Y_0, Y_1),$$

and thus the proof follows in a similar way as the proof of Corollary 3.5. □

We give now an applications to operators from $C(K)$-spaces into Banach spaces of finite cotype. Recall that a Banach space X is said to be of cotype q $(2 \leqslant q < \infty)$ if there exists a constant $C > 0$ so that for all $n \in \mathbb{N}$ and any $x_1, \ldots, x_n \in X$

the inequality

$$\left(\sum_{i=1}^{n}\|x_i\|_X^q\right)^{1/q} \leqslant C \operatorname*{Ave}_{\varepsilon_i=\pm 1}\left\|\sum_{i=1}^{n}\varepsilon_i x_i\right\|_X$$

holds. We need the following result (see [12, Theorem 11.14]).

Theorem 3.7. *Suppose that Y is a Banach space of cotype q, where $2 \leqslant q < \infty$, and that K is a compact Hausdorff space. Then we have*:

(a) *If $q = 2$, then $\mathcal{L}(C(K), Y) = \Pi_2(C(K), Y)$.*
(b) *If $2 < q < \infty$, then $\mathcal{L}(C(K), Y) = \Pi_{q,p}(C(K), Y)$ for all $p < q$.*

Theorem 3.8. *Let (X_0, X_1) be a Banach couple such that X_j has cotype q_j, $j = 0, 1$. If X is a Banach space such that $X_0 \cap X_1 \hookrightarrow X$ and φ is the characteristic function of X with respect to (X_0, X_1), then for every operator $T : (C(K), C(K)) \to (X_0, X_1)$ we have*

(i) *$T : C(K) \to X$ is $(\ell_\psi, 2)$-summing, where $\psi^{-1}(t) \asymp \overline{\varphi}(t^{1/q_0}, t^{1/q_1})$.*
(ii) *$x_n(T : C(K) \to X) \prec \overline{\varphi}(n^{-1/q_0}, n^{-1/q_1})$ for each $n \in \mathbb{N}$.*

Proof. (i) Theorem 3.7 ensures that $T : C(K) \to X_j$ is $(q_j, 2)$-summing, and thus Theorem 3.4 implies that $T : C(K) \to X$ is $(\phi(\ell_{q_0}, \ell_{q_1}), 2)$-summing, where ϕ is a concave minorant of $\overline{\varphi}$. Since $\phi(\ell_{q_0}, \ell_{q_1})$ coincides up to equivalence of norms with an Orlicz sequence space ℓ_ψ with

$$\psi^{-1}(t) \asymp \overline{\varphi}\big(t^{1/q_0}, t^{1/q_1}\big),$$

the statement follows.

(ii) The fundamental function of an Orlicz sequence space ℓ_ψ equals

$$\lambda_{\ell_\psi}(n) = \frac{1}{\psi^{-1}(1/n)}, \quad n \in \mathbb{N}.$$

Thus the required estimate follows by Proposition 2.2(i) with $G = \ell_\psi = \phi(\ell_{q_0}, \ell_{q_1})$. $\square$

Below we determine – for a large class of r.i. spaces $X(\mu)$ on a finite non-atomic measure space (Ω, μ) – the asymptotic behavior of the Weyl numbers of the inclusion map $I : L_\infty(\mu) \to X(\mu)$. For any $f \in L_0(\mu)$ we define its decreasing rearrangement f^* on $(0, \mu(\Omega))$ by $f^*(t) = \inf\{\lambda > 0;\ \mu(|f| > \lambda) \leqslant t\}$. Now let X be a quasi-Banach lattice on $(0, a)$ $(0 < a \leqslant \infty)$ with Lebesgue measure. We say that X is a *rearrangement invariant* (r.i.) quasi-Banach space if $\|f\|_X = \|f^*\|_X$ for all $f \in X$. If (Ω, μ) is an arbitrary measure space and X is an r.i. quasi-Banach space on $(0, \mu(\Omega))$, then we define the r.i. space $X(\mu)$ on (Ω, μ) to be the space of all $f \in L_0(\mu)$ such that $f^* \in X$ with $\|f\|_{X(\mu)} = \|f^*\|_X$. In what follows we use for r.i. Banach spaces the shorter notation *r.i. spaces*.

Let $X(\mu)$ be an r.i. space on (Ω, μ) determined by an r.i. space X on $(0, a)$ with $a = \mu(\Omega)$. The *fundamental function* of X is defined by $\psi_X(t) = \|\chi_{(0,t)}\|_X$ for $t \in (0, a)$, where χ_A denotes the characteristic function of the set A. For a given $t > 0$ the *dilation operator* D_t is defined by $D_t x(s) = x(s/t)\chi_{(0,a)}(s/t)$, $s \in (0, a)$. D_t is bounded in every r.i. space X with $\|D_t\| \leqslant \max(1, t)$. The *lower* and the *upper Boyd indices* are defined by

$$\alpha_X = \lim_{t\to 0+}\frac{\ln\|D_t\|}{\ln t}, \qquad \beta_X = \lim_{t\to\infty}\frac{\ln\|D_t\|}{\ln t},$$

respectively. In general, $0 \leqslant \alpha_X \leqslant \beta_X \leqslant 1$.

The *lower* and *upper dilation indices* of a quasi-concave function $\psi : [0, a) \to [0, \infty)$ are defined by

$$\gamma_\psi = \lim_{t\to 0+}\frac{\ln\overline{\psi}(t)}{\ln t}, \qquad \overline{\psi} = \lim_{t\to\infty}\frac{\ln\overline{\psi}(t)}{\ln t},$$

respectively, where $\overline{\psi}(t) = \sup_{0<s<a,\,0<st<a}\frac{\psi(st)}{\psi(t)}$. We have $0 \leqslant \gamma_\psi \leqslant \delta_\psi \leqslant 1$, and since $\overline{\psi}_X(t) \leqslant \|D_t\|_X$ for every $t > 0$, it follows that $\alpha_X \leqslant p_X \leqslant q_X \leqslant \beta_X$, where $p_X := \gamma_{\psi_X}$ and $q_X := \delta_{\psi_X}$. For general properties of r.i. spaces and indices we refer to [1,18].

We first prove the following lemma which extends the result [17, Corollary 3.a.10] for the inclusion map $I : L_\infty(\mu) \to L_{p,1}(\mu)$.

Lemma 3.9. *Let $X(\mu)$ be a rearrangement invariant space on a non-atomic probability space (Ω, μ). Then we have*

(i) $(2\sqrt{2}n\int_0^{1/2n}\psi_X(t)^{-1}\,dt)^{-1} \leqslant x_n(I : L_\infty(\mu) \to X(\mu))$ *for each* $n \in \mathbb{N}$.
(ii) *If* $q_X < 1$, *then* $\psi_X(1/n) \prec x_n(I : L_\infty(\mu) \to X(\mu))$.
(iii) *If* $X(\mu)$ *is an interpolation space with respect to* $(L_2(\mu), L_\infty(\mu))$, *then*

$$\psi_X(1/n) \prec x_n\big(I : L_\infty(\mu) \to X(\mu)\big) \prec \overline{\psi}_X(1/n).$$

Proof. (i) For a given positive integer m we choose disjoint measurable sets E_k such that $\mu(E_k) = 1/m$ for $k = 1, \ldots, m$. Define operators $J_m : \ell_\infty^m \to L_\infty(\mu)$ and $Q_m : X(\mu) \to \ell_\infty^m$ by

$$J_m(\xi) := \sum_{k=1}^m \xi_k \chi_{E_k}, \quad \text{for } \xi = \{\xi_k\} \in \ell_\infty^m,$$

and

$$Q_m(f) := \sum_{k=1}^m \bigg(\int_{E_k} x(s)\,d\mu(s)\bigg) e_k, \quad \text{for } x \in X(\mu).$$

We have

$$\big\| J_m : \ell_\infty^m \to L_\infty(\mu)\big\| = 1.$$

Since $\|x\|_{X(\mu)} \geqslant \|x^*\chi_{(0,t)}\|_X \geqslant x^*(t)\psi_X(t)$ for every $x \in X(\mu)$ and $t \in (0,1)$, we have

$$\int_{E_k} |x|\,d\mu \leqslant \int_0^{1/m} x^*(t)\,dt \leqslant \|x\|_{X(\mu)} \int_0^{1/m} \frac{1}{\psi_X(t)}\,dt,$$

and whence

$$\big\| Q_m : X(\mu) \to \ell_\infty^m\big\| \leqslant \int_0^{1/m} \frac{1}{\psi_X(t)}\,dt.$$

Using the following obvious factorization of the identity map $I_m : \ell_\infty^m \to \ell_\infty^m$,

$$I_m : \ell_\infty^m \xrightarrow{J_m} L_\infty(\mu) \xrightarrow{mI} X(\mu) \xrightarrow{Q_m} \ell_\infty^m$$

and the Stechkin formula (see [25, Prop. 2.9.11])

$$a_n\big(I : \ell_1^m \to \ell_2^m\big) = \left(\frac{m-n+1}{m}\right)^{1/2}, \quad \text{for } n = 1, \ldots, m,$$

we conclude that

$$\begin{aligned}
\left(\frac{m-n+1}{m}\right)^{1/2} &= a_n\big(I : \ell_1^m \to \ell_2^m\big) = a_n\big(I : \ell_2^m \to \ell_\infty^m\big) \\
&= x_n\big(I : \ell_2^m \to \ell_\infty^m\big) \leqslant x_n\big(I : \ell_\infty^m \to \ell_\infty^m\big) \\
&\leqslant m\|Q_m\| x_n\big(I : L_\infty(\mu) \to X(\mu)\big)\|J_m\| \\
&\leqslant m\left(\int_0^{1/m} \frac{1}{\psi_X(t)}\,dt\right) \cdot x_n\big(I : L_\infty(\mu) \to X(\mu)\big).
\end{aligned}$$

Setting $m = 2n$ yields the required estimate.

(ii) Since $q_X < 1$, $\gamma_\psi > 0$ where $\psi(t) = t/\psi_X(t)$ for $t \in (0,1)$. This implies (see [18, Corollary 3, p. 80]) that

$$\int_0^t \frac{\psi(s)}{s}\,ds \prec \psi(t), \quad t \in (0,1).$$

Combining this estimate with the fact that ψ is non-decreasing, we obtain the required estimate by (i).

(iii) Our hypothesis on X yields that $\psi_X(t) \asymp \varphi(\sqrt{t}, 1)$ (see [22]), where φ is the characteristic function of X with respect to (L_2, L_∞). This implies that

$$\overline{\psi}_X(t) \asymp \overline{\varphi}(\sqrt{t}, 1).$$

Combining this equivalence with the fact that every $L_\infty(\mu)$-space is isometric to a $C(K)$-space and $L_2(\mu)$ is a 2-concave Banach lattice, we obtain from Theorem 3.8(ii)

$$x_n\big(I : L_\infty(\mu) \to X(\mu)\big) \prec \overline{\psi}_X(1/n).$$

Now observe that our hypothesis on X implies that $\beta_X \leqslant 1/2$, and whence $q_X \leqslant 1/2$. Thus (ii) applies and gives the lower estimate. $\square$

We remark that Lorentz–Shimogaki showed that (L_p, L_∞) is a Calderón couple for any measure space (for details see [2,24]). Since for every $f \in L_p + L_\infty$ and $t > 0$,

$$\left(\int_0^t f^*(s)^p\, ds\right)^{1/p} \leqslant K\big(t^{1/p}, f; L_p, L_\infty\big) \leqslant 2^{1-1/p}\left(\int_0^t f^*(s)^p\, ds\right)^{1/p}$$

we conclude that X is an interpolation space with respect to (L_p, L_∞) if and only if the following holds: If $f \in L^0(\mu)$ and $g \in X$ satisfy

$$\int_0^t f^*(s)^p\, ds \leqslant \int_0^t g^*(s)^p\, ds,$$

then this implies $f \in X$.

Using the above description it is easy to verify that there is a large class of natural r.i. spaces which are interpolation spaces with respect to (L_2, L_∞), e.g. 2-convex spaces or spaces with $\beta_X < 1/2$. In particular, all Lorentz space $L_{p,q}$ with $2 < p < \infty$ and $1 \leqslant q \leqslant \infty$ are of this type.

4. Concavity constants with few vectors

As we have seen in the previous sections, estimates of concavity constants with respect to n vectors are very useful in the study of Weyl numbers. In this section we will provide estimates of these constants for some special classes of rearrangement invariant spaces, in particular, we will consider certain Lorentz and Orlicz spaces. For this purpose we need some results due to Kalton in the memoir [14], where the uniqueness of lattice structure on Banach spaces is studied. In [14] the author discusses the question of finding a condition on a Banach lattice such that it does not contain uniformly complemented ℓ_2^n's. To this aim, for a given Banach lattice X he defined $d_n = d_n(X)$ to be the least constant such that for disjoint $x_1, \dots, x_n \in X$ we have

$$\sum_{k=1}^n \|x_k\|_X \leqslant d_n \left\|\sum_{k=1}^n x_n\right\|_X$$

and $e_n = e_n(X)$ to be the least constant so that for disjoint $x_1, \dots, x_n$ we have

$$\left\|\sum_{k=1}^n x_n\right\|_X \leqslant e_n \max_{1\leqslant k\leqslant n} \|x_k\|_X.$$

Later on we will use the following result. Parts (i) and (ii) follow from the proof of Proposition 9.2 in [14] presented there for abstract separable order-continuous Banach lattices.

In terms of concavity constants, the first formula in (ii) means $M^n_{(1)}(X) \leqslant d_n(X)$. Since the reverse inequality is obvious by definition of these constants, we have $M^n_{(1)}(X) = d_n(X)$. Part (iii) is a simple consequence of (ii) and the formula $M^n_{(p)}(X_{(p)}) = M^n_{(1)}(X)^{1/p}$ for the p-convexification procedure with $1 < p < \infty$.

Proposition 4.1. *Let X be an order-continuous Banach lattice on (Ω, μ).*

(i) *For the Köthe dual space X' of X we have $d_n(X') = e_n(X)$ and $e_n(X') = d_n(X)$.*

(ii) *If $x_1, \dots, x_n$ are any elements in X, then*

$$\sum_{k=1}^{n} \|x_k\|_X \leqslant d_n \left\| \sum_{k=1}^{n} |x_n| \right\|_X \quad \text{and} \quad \left\| \max_{1\leqslant k\leqslant n} |x_k| \right\|_X \leqslant e_n \max_{1\leqslant k\leqslant n} \|x_k\|_X.$$

(iii) *For each $n \in \mathbb{N}$ and every $1 \leqslant p < \infty$ we have $M_{(p)}^n(X_{(p)}) = d_n(X)^{1/p}$.*

Our next aim is to determine concavity constants with respect to n vectors for quite general Lorentz spaces. Similar results for usual concavity constants have been obtained by Kamińska and Parrish in [15].

Let (Ω, μ) be an arbitrary measure space, and let $\psi : [0, \mu(\Omega)) \to [0, \infty)$ be a concave function with $\psi(0+) = 0$. The *Lorentz space* $\Lambda_p(\psi)$, $1 \leqslant p < \infty$, is defined as the set of all $f \in L_0(\mu)$ such that

$$\|f\|_{\Lambda_p(\psi)} := \left(\int_0^{\mu(\Omega)} f^*(s)^p \, d\psi(s) \right)^{1/p} < \infty.$$

The *Marcinkiewicz space* $M(\psi)$ consists of all $f \in L_0(\mu)$ such that

$$\|f\|_{M(\psi)} := \sup_{0<t<\mu(\Omega)} \frac{\int_0^t f^*(s)\, ds}{\psi(t)} < \infty.$$

The spaces $\Lambda_p(\psi)$ and $M(\psi)$ are r.i. Banach function spaces equipped with the norms defined above. If $p = 1$ we write $\Lambda(\psi)$ instead of $\Lambda_1(\psi)$. Moreover the Köthe duality $\Lambda(\psi)' = M(\psi)$ with equality of norms can be obtained in a similar way as the corresponding result on the description of the Banach dual space of the Lorentz space $\Lambda(\psi)$ under the condition $\lim_{t\to\infty} \psi(t) = \infty$ whenever $\mu(\Omega) = \infty$ (cf. [18]). If $\psi(t) = t^{p/q}$ with $1 \leqslant p < q < \infty$, we recover the usual Lorentz spaces, $\Lambda_p(\psi) = L_{q,p}$, and for $\psi(t) = t^{1/q}$ we obtain $M(\psi) = L_{q',\infty}$, where $1/q + 1/q' = 1$.

Lemma 4.2. *Let (Ω, μ) be a non-atomic measure space and let $\psi : [0, \mu(\Omega)) \to [0, \infty)$ be a concave function with $\psi(0+) = 0$. If $\mu(\Omega) = \infty$ we assume also $\lim_{t\to\infty} \psi(t) = \infty$. Then for the Lorentz space $\Lambda_p(\psi)$ on (Ω, μ) with $1 \leqslant p < \infty$ we have*

$$M_{(p)}^n\big(\Lambda_p(\psi)\big) = \left(\sup_{0<t<\mu(\Omega)} \frac{n\psi(t/n)}{\psi(t)} \right)^{1/p}.$$

In particular, $M_{(p)}^n(\Lambda_p(\psi)) \asymp n^{1/p} \psi^{1/p}(1/n)$ whenever ψ is submultiplicative at 0.

Proof. Let $\varphi(n) := \sup_{0<t<\mu(\Omega)} \frac{n\psi(t/n)}{\psi(t)}$ for all positive integers n. It was shown in [22] that

$$e_n\big(M(\psi)\big) = \varphi(n).$$

It is easy to check that our hypotheses on ψ imply that $\Lambda(\psi)$ is order-continuous. Since $\Lambda(\psi)' = M(\psi)$ with equality of norms, Proposition 4.1 implies for each positive integer n

$$M_{(p)}^n\big(\Lambda_p(\psi)\big) = d_n\big(\Lambda(\psi)\big)^{1/p} = e_n\big(M(\psi)\big)^{1/p} = \varphi(n)^{1/p},$$

and this completes the proof. □

Corollary 4.3. *Let ϕ be an Orlicz function and let $\psi(t) = t\phi^{-1}(\ln(1 + 1/t))$ for $t > 0$. For the Lorentz space $\Lambda_p(\psi)$ on a finite non-atomic measure space (Ω, μ) with $1 \leqslant p < \infty$ we have*

$$M_{(p)}^n\big(\Lambda_p(\psi)\big) \asymp \phi^{-1}\big(\ln(n + 1)\big)^{1/p}.$$

Proof. Since $M_{(p)}^n(\Lambda_p(\psi)) = d_n(\Lambda(\psi))^{1/p}$, the assertion is a consequence of the equivalence

$$d_n\big(\Lambda(\psi)\big) = \varphi(n) := \sup_{0<t<\mu(\Omega)} \frac{n\psi(t/n)}{\psi(t)} \asymp \phi^{-1}\big(\ln(n + 1)\big)$$

proved in [22]. For completeness, we include a proof here.

Since ϕ^{-1} is a concave function on $[0, \infty)$, we have $\phi^{-1}(u + v) \leqslant \phi^{-1}(u) + \phi^{-1}(v)$ for all $u, v > 0$. It follows that

$$\phi^{-1}\big(\ln(1 + n/t)\big) \leqslant \phi^{-1}\big(\ln(n + 1)\big) + \phi^{-1}\big(\ln(1 + 1/t)\big),$$

for every $0 < t < \mu(\Omega)$. This implies (by $\mu(\Omega) < \infty$)

$$\varphi(n) = \sup_{0<t<\mu(\Omega)} \frac{\phi^{-1}(\ln(1+n/t))}{\phi^{-1}(\ln(1+1/t))} \asymp \phi^{-1}\big(\ln(n+1)\big)$$

and gives the desired equivalence. $\square$

We note that for $\phi(t) = t^\alpha$ $(\alpha > 1)$ we obtain the well-known Zygmund spaces $L(\log L)^{1/\alpha}$ which are special cases of Orlicz spaces (see [1]).

5. Eigenvalues of certain integral operators

In this final section we apply our abstract results to integral operators with kernels of weakly singular type. We obtain upper estimates for the eigenvalues of such operators and, using appropriate examples, we also show the optimality of these estimates.

The basic notion in the context of eigenvalue distributions of operators in Banach spaces is that of a Riesz operator; for the precise definition we refer to the monographs [17,25,5]. The most important examples of Riesz operators are power-compact and so in particular compact operators acting in a complex Banach space. The main property of a Riesz operator $T : X \to X$ is that its spectrum has no non-zero accumulation points and consists of eigenvalues of finite multiplicity only (except possibly zero). Therefore one can arrange all eigenvalues of T in a sequence $(\lambda_k(T))$ such that

$$\big|\lambda_1(T)\big| \geqslant \big|\lambda_2(T)\big| \geqslant \cdots \geqslant 0$$

and each eigenvalue is counted according to its algebraic multiplicity. If T has less than n eigenvalues $\lambda \neq 0$, then we put $\lambda_n(T) = \lambda_{n+1}(T) = \cdots = 0$.

For a large class of differential operators on bounded domains $\Omega \subset \mathbb{R}^N$ the inverse operators are integral operators with the so-called weakly singular kernels, i.e. measurable functions $k : \Omega^2 \setminus \Delta \to \mathbb{C}$, where $\Delta := \{(x,x);\ x \in \Omega\}$, defined by

$$k(x,y) = \frac{l(x,y)}{|x-y|^{N-\alpha}} \quad \text{with } 0 < \alpha < N \text{ and } l \in L_\infty\big(\Omega^2\big). \tag{1}$$

Here $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^N$. It was shown in [16] (see also [17, Theorem 3.a.11]) that the integral operator T_k is a Riesz operator in any $L_r(\Omega)$-space with $1 \leqslant r \leqslant \infty$, that its eigenvalues are square summable whenever $\alpha > N/2$, and satisfy otherwise

$$\big|\lambda_n(T_k)\big| \prec \begin{cases} n^{-\alpha/N}, & \text{if } 0 < \alpha < N/2, \\ (\frac{\ln(n+1)}{n})^{1/2}, & \text{if } \alpha = N/2. \end{cases}$$

These estimates are asymptotically best possible. The degree of singularity of these operators is expressed by a single parameter α, $0 < \alpha < N$.

Later on, several variants of weakly singular kernels have been investigated, for instance when the singularity involves an additional logarithmic term,

$$k(x,y) = \frac{l(x,y)(1 + |\log|x-y||)^\gamma}{|x-y|^{N-\alpha}} \quad \text{with } \gamma \in \mathbb{R}. \tag{2}$$

Here the degree of singularity is described by two parameters, α and γ. In the proofs of the corresponding eigenvalue results for these operators different methods have been used; entropy and interpolation techniques in [3,4,7] and classical Hilbert space methods in [8,6].

In order to find a suitable framework for generalizing weakly singular integral operators, let us come back to kernels of the form (1). Set $p = N/\alpha$ and consider the Lorentz space $X = L_{p,1}$, whose Köthe dual space is $X' = L_{p',\infty}$ (all spaces are defined on a bounded subset $\Omega \subset \mathbb{R}^N$). Note that the kernel k belongs to the *mixed space* $L_\infty[X']$, which means that $g(x) := \|k(x,\cdot)\|_{L_{p',\infty}}$ is measurable and bounded. Similarly we can argue for kernels of the form (2) if we choose the space X appropriately.

Now let X be any r.i. Banach function space defined on a finite measure space. Then we call a kernel $k \in L_\infty[X']$ of *weakly singular type*. The degree of singularity is measured by the space X, or its Köthe dual X'. Clearly the integral operator $T_k : X \to L_\infty$ and the identity $I : L_\infty \hookrightarrow X$ are bounded, thus we have the factorization $L_\infty \overset{I}{\hookrightarrow} X \xrightarrow{T_k} L_\infty$. Alternatively we could regard T_k as operator in any space Y with $L_\infty \hookrightarrow Y \hookrightarrow X$. If T_k is a Riesz operator on L_∞, then the principle of related operators (see [25]) shows that $T_k : Y \to Y$ is Riesz, too, and its eigenvalue are the same as those of $T_k : L_\infty \to L_\infty$, with the same algebraic multiplicities.

We can now state the following applications of our previous results.

Theorem 5.1. *Let X be an r.i. Banach function space on a non-atomic finite measure space (Ω, μ), and assume that X is an interpolation space with respect to $(L_2(\mu), L_\infty(\mu))$. Then every kernel $k \in L_\infty[X']$ generates a Riesz operator T_k in L_∞ (or in any space Y with $L_\infty \hookrightarrow Y \hookrightarrow X$) with eigenvalues behaving like*

$$\left|\lambda_n(T_k)\right| = \mathcal{O}\left(\overline{\psi}_X(1/n)\right).$$

Proof. By the factorization $L_\infty \overset{I}{\hookrightarrow} X \xrightarrow{T_k} L_\infty$ and part (iii) of Lemma 3.9 we have

$$x_n(T_k : L_\infty \to L_\infty) \leqslant x_n(I : L_\infty \to X)\|T_k : X \to L_\infty\| \prec \overline{\psi}_X(1/n).$$

Now recall the known estimate (see [17, Theorem 2.a.4])

$$\left(\prod_{j=1}^{n}\left|\lambda_j(T)\right|\right)^{1/n} \leqslant \sqrt{2e}\left(\prod_{j=1}^{n}\dot{x}_j(T)\right)^{1/n}$$

for all Riesz operators $T : Y \to Y$ in any complex Banach space Y, where $(\dot{x}_j(T))_{j\in\mathbb{N}}$ denotes the doubled sequence $(x_1(T), x_1(T), x_2(T), x_2(T), \ldots)$. Combining these two estimates we get for even n

$$\left|\lambda_n(T)\right| \leqslant \sqrt{2e}\left(\prod_{j=1}^{n/2}\overline{\psi}_X(1/j)^2\right)^{1/n}.$$

Since $\overline{\psi}_X$ is non-decreasing and satisfies a Δ_2-condition, there exist constants $\alpha > 0$ and $C_\alpha > 0$ such that

$$\overline{\psi}_X(t) \leqslant C_\alpha\left(\frac{t}{s}\right)^{\alpha}\overline{\psi}_X(s), \quad \text{for every } 0 < s < t \leqslant 1.$$

By Stirling's formula we have $(m!)^{1/m} \asymp m$, and this implies

$$\left(\prod_{j=1}^{n/2}\overline{\psi}_X(1/j)^2\right)^{1/n} \leqslant C_\alpha\overline{\psi}_X(1/n)\left(\prod_{j=1}^{n/2}(n/j)^{2\alpha}\right)^{1/n} \asymp \overline{\psi}_X(1/n).$$

Combining the above estimates yields the assertion. □

An immediate consequence of this theorem is the following corollary; as explained above it extends the result for the classical weakly singular integral operators to a larger kernel class.

Corollary 5.2. *Let $2 < p < \infty$ and assume that the Lorentz space $L_{p,1}$ is defined over some non-atomic finite measure space. Then every kernel $k \in L_\infty[L_{p',\infty}]$ generates a Riesz operator T_k in $L_{p,1}$ (or in any L_r with $p < r \leqslant \infty$) with eigenvalues behaving like*

$$\left|\lambda_n(T_k)\right| = \mathcal{O}\left(n^{-1/p}\right).$$

Proof. The space $X = L_{p,1}$ is an r.i. Banach function space, in addition it is an interpolation space with respect to the couple (L_2, L_∞). The Köthe dual is $X' = L_{p',\infty}$ and for the fundamental function one has $\psi_X(t) \asymp t^{1/p} \asymp \overline{\psi}_X(t)$. Since the measure is finite, we have $L_\infty \hookrightarrow L_r \hookrightarrow L_{p,1}$. Thus all assumptions of the preceding theorem are satisfied, and the assertion follows. □

Now let us turn to integral operators of weakly singular type in more general Lorentz spaces $\Lambda(\psi)$. Our aim is to prove upper eigenvalue estimates, and to show that the obtained bounds are optimal within the considered kernel class.

In the remaining part of the paper the underlying measure space of all function spaces will be the interval $(0, 1)$ equipped with Lebesgue measure, and ψ denotes always a concave function $\psi : (0, 1) \to (0, \infty)$ with $\psi(0+) = 0$. Since the fundamental function of the Lorentz space $\Lambda(\psi)$ equals ψ, the following result is an immediate consequence of Theorem 5.1.

Corollary 5.3. *Assume that the Lorentz space $\Lambda(\psi)$, defined over a finite non-atomic measure space, is an interpolation space with respect to the couple $(L_2(0, 1), L_\infty(0, 1))$. Then every kernel $k \in L_\infty[M(\psi)]$ generates a Riesz operator T_k in L_∞ (or in any space Y with $L_\infty \hookrightarrow Y \hookrightarrow \Lambda(\psi)$) with eigenvalues satisfying*

$$\left|\lambda_n(T_k)\right| = \mathcal{O}\left(\overline{\psi}(1/n)\right).$$

Finally, in order to illustrate the optimality of the eigenvalue results in Theorem 5.1 and Corollary 5.3, we specify even more and consider Lorentz spaces $\Lambda(\psi)$ with

$$\psi(t) = \psi_{a,b}(t) := t^a(1-\log t)^b, \quad \text{where } 0<a<1 \text{ and } b \geqslant 0.$$

Clearly, if $b=0$ and $a=1/p$, then $\Lambda(\psi_{a,b}) = L_{p,1}$ and $M(\psi_{a,b}) = L_{p',\infty}$.

The norms of the dilation operators D_t with $t \geqslant 1$ can easily be estimated by $\|D_t : \Lambda(\psi_{a,b}) \to \Lambda(\psi_{a,b})\| \leqslant t^a$, whence the upper Boyd index is

$$\beta_{\Lambda(\psi_{a,b})} = \lim_{t\to\infty} \frac{\ln\|D_t\|}{\ln t} \leqslant a.$$

(In fact we have even $\beta_{\Lambda(\psi_{a,b})} = a$.) In view of the remark after Lemma 3.9, we conclude that $\Lambda(\psi_{a,b})$ is an interpolation space with respect to (L_2, L_∞) whenever $0<a<1/2$. Moreover we have $\psi_{a,b} \asymp \overline{\psi}_{a,b}$.

From these observations and Corollary 5.3 we immediately obtain the following result.

Corollary 5.4. *Let $0<a<1/2$ and $b \geqslant 0$. Then every kernel $k \in L_\infty[M(\psi_{a,b})]$ generates a Riesz operator T_k in L_∞ (or in any space Y with $L_\infty \hookrightarrow Y \hookrightarrow \Lambda(\psi_{a,b})$) with eigenvalues*

$$\left|\lambda_n(T_k)\right| = \mathcal{O}\left(n^{-a}(1+\log n)^b\right).$$

Finally we show the optimality of the estimate in Corollary 5.4 – and therefore also the optimality of Theorem 5.1 and Corollaries 5.2 and 5.3 – by using convolution operators. Let $g \in L_1(0,1)$ and $k(x,y) := \tilde{g}(x-y)$, where $\tilde{g} : \mathbb{R} \to \mathbb{R}$ denotes the 1-periodic extension of g. Then it is well known and easy to verify that the Fourier coefficients of g,

$$\hat{g}(n) := \int_0^1 g(x)e^{-2\pi inx}\,dx, \quad n \in \mathbb{Z},$$

are eigenvalues of the integral operator $T_k : L_\infty \to L_\infty$ with corresponding eigenfunctions $e^{2\pi inx}$. If $g \in X'$ for some r.i. Banach function space over the interval $(0,1)$, then obviously we have $k \in L_\infty[X']$.

For the construction of our examples we need the following lemma which is certainly known, but for the convenience of the reader we give the proof. It is based on Abel's summation as used e.g. in the classical paper [20] by Lorentz.

Lemma 5.5. *Let $0<a<1$ and $b \geqslant 0$. Then the series*

$$\sum_{n=1}^\infty \frac{(1+\log n)^b}{n^a} e^{2\pi inx}$$

converges for all $x \in \mathbb{R}\setminus\mathbb{Z}$, and its sum function belongs to the Marcinkiewicz space $M(\psi_{a,b})$.

Proof. Clearly there is an index $n_0 = n_0(a,b)$ such that the sequence $a_n = n^{-a}(1+\log n)^b$ is decreasing for $n \geqslant n_0$. Further, it is easy to verify that for $0<|x| \leqslant 1/2$ the inequality

$$B(x) := \sup_{M>N} \left|\sum_{n=N}^M e^{2\pi inx}\right| \leqslant \frac{1}{|\sin \pi x|} \leqslant \frac{2}{|x|}$$

holds. Since $\lim_{n\to\infty} a_n = 0$ and a_n is decreasing for $n \geqslant n_0$, Abel's summation applies and shows that the series $\sum_{n=1}^\infty a_n e^{2\pi inx}$ converges for all x with $0<|x| \leqslant 1/2$, and by the 1-periodicity for all $x \in \mathbb{R}\setminus\mathbb{Z}$. Let now $N > n_0$ and $\frac{1}{N} \leqslant |x| \leqslant \frac{1}{N-1}$. We estimate the sum function, say g, of the above series by

$$\left|g(x)\right| \leqslant \sum_{n=1}^{N-1} a_n + \left|\sum_{n=N}^\infty a_n e^{2\pi inx}\right|,$$

where we have for the first sum

$$\sum_{n=1}^{N-1} a_n \asymp Na_N = N^{1-a}(1+\log N)^b \asymp x^{a-1}(1-\log x)^b,$$

while the second sum can be estimated as follows:

$$\left|\sum_{n=N}^\infty a_n e^{2\pi inx}\right| \leqslant a_N B(x) \leqslant a_N \frac{2}{|x|} \asymp x^{a-1}(1-\log x)^b.$$

This gives $|g(x)| \leqslant c_1 x^{a-1}(1-\log x)^b$ for $0 < |x| \leqslant 1/n_0$, and whence

$$\left|g^*(s)\right| \leqslant c_2 s^{a-1}(1-\log s)^b, \quad \text{for } 0 < s \leqslant 1,$$

where the constants $c_1, c_2 > 0$ depend only on a and b. Consequently we get

$$\int_0^t g^*(s)\,ds \leqslant c_3 t^a (1-\log t)^b = c_3 \psi_{a,b}(t), \quad \text{for } 0 < t \leqslant 1,$$

and this proves $g \in M(\psi_{a,b})$. $\square$

Now we are prepared to construct the examples that will show the optimality of the upper eigenvalue estimate in Corollary 5.4. Let g be the function from the preceding lemma. Then the convolution kernel $k(x, y) = g(x-y)$, $x, y \in (0,1)$, belongs to $L_\infty[M(\psi_{a,b})]$, and the eigenvalues of $T_k : L_\infty \to L_\infty$ are

$$\lambda_n(T_k) = \hat{g}(n) = n^{-a}(1+\log n)^b, \quad n \in \mathbb{N},$$

which matches the upper bound in Corollary 5.4. This shows that Corollary 5.4, and therefore also all other eigenvalue estimates in this section, are asymptotically optimal.

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